Exact Solutions of Boussinesq Equation

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Joint work with: L. Kaur, K. Sakkaravarthi, R. Sakthivel, K. Murugesan, arxiv:2004.09460 (2020)

International Conference on Generalized Functions, August 31-September 4, 2020 Ghent, Belgium

Objective

- **To construct rogue wave solutions of a new integrable (2+1)-dimensional Boussinesq equation by using Bell polynomial, Hirota's bilinearization and a generalized polynomial function.**

- **To explore the dynamics of these localized structures and manipulation of their identities by tuning arbitrary parameters.**

Introduction

- Model: (2+1)D integrable Boussinesq equation governing the gravity waves and collisions of surface water waves proposed recently [1]:

$$u_{tt} - u_{xx} - \beta(u^2)_{xx} - \gamma u_{xxxx} + \frac{\alpha^2}{4}u_{yy} + \alpha u_{yt} = 0, \tag{1}$$

  where α, β and γ are nonzero constants.
- Equation (1) shall reduces to different versions of Boussinesq and Benjamin-Ono type models for suitable choices of α, β and γ.
- Results on solitary waves, rational solutions, periodic and lump solutions to various types of (2+1)D and (1+1)D Boussinesq equations are reported already.

Rouge Waves

Mathematically, rogue wave is defined as the space and time localised solution consist of a background and bulging part of an nonlinear evolution equation.

The rogue wave solution for the focusing nonlinear Scrodinger equation (NLSE)

$$iq_t + q_{xx} + 2(|q|^2 - 1)q = 0$$

is

$$q(x,t) = 1 - \frac{4(1+4it)}{1+4x^2+16t^2}$$

.

The above solution function have the property as mentioned in the definition, was discovered by H.Peregrine (1983).

- Appears from nowhere and disappears without trace.
- $|q(x,t)| \to 1$ as $|x| + |t| \to \infty$
- Its magnitude is generally 3 to 6 times. i.e., at center it is magnified $3 \leq |q(0,0)| \leq 6$.

Methodology

The Hirota bilinear method and Bell polynomial are used.

$$\text{Bilinear transformation:} u(x,\tau) = \frac{3\gamma}{\beta} q_{xx} + u_0, \tag{2}$$

where $\tau = y + a_1 t,\ x = x$, and $q = 2\ln(\mathcal{R}(x,\tau))$.
Billinear form of Eq. (1):

$$\left((a_1^2 + \frac{\alpha^2}{4} + \alpha a_1)D_\tau^2 - (2\beta u_0 + 1)D_x^2 - \gamma D_x^4\right)\mathcal{R}\cdot\mathcal{R} = 0. \tag{3}$$

Utilize the following generalized ansatz [2, 3]:

$$\mathcal{R}_{r+1}(x,\tau;\lambda,\mu) = R_{r+1} + 2\lambda\tau F_r + 2\mu x G_r + (\lambda^2+\mu^2)R_{r-1}, \tag{4a}$$

$$\text{with} \quad R_r(x,\tau) = \sum_{n=0}^{r(r+1)/2}\sum_{i=0}^{n} c_{r(r+1)-2n,2i}\, x^{r(r+1)-2n}\tau^{2i}, \tag{4b}$$

$$F_r(x,\tau) = \sum_{n=0}^{r(r+1)/2}\sum_{i=0}^{n} e_{r(r+1)-2n,2i}\, x^{r(r+1)-2n}\tau^{2i}, \tag{4c}$$

$$G_r(x,\tau) = \sum_{n=0}^{r(r+1)/2}\sum_{i=0}^{n} h_{r(r+1)-2n,2i}\, x^{r(r+1)-2n}\tau^{2i}, \tag{4d}$$

where $\lambda, \mu, c_{p,q}, e_{p,q}$ and $h_{p,q}$ are arbitrary real parameters.

First Order Rogue Wave

First order rogue wave for $r = 0$ in (4a) [4]:

$$\mathcal{R} = \mathcal{R}_1(x,\tau) = c_{0,0} + c_{0,2}\tau^2 + c_{2,0}x^2. \tag{5}$$

$$u = u_0 + \frac{12\gamma}{\beta}\left(\frac{\frac{-3\gamma}{(1+2u_0\beta)} - (x-\lambda)^2 - \frac{4(1+2u_0\beta)}{(2a_1+\alpha)^2}(y+a_1t-\mu)^2}{\left(\frac{-3\gamma}{(1+2u_0\beta)} + (x-\lambda)^2 - \frac{4(1+2u_0\beta)}{(2a_1+\alpha)^2}(y+a_1t-\mu)^2\right)^2}\right). \tag{6}$$

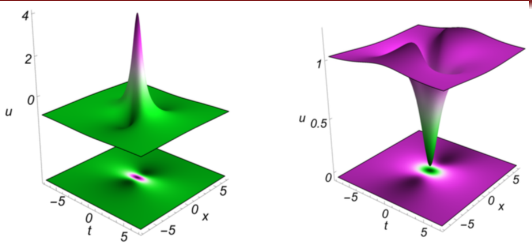

Figure 1: Doubly localized bright and dark rogue waves in $x - t$.

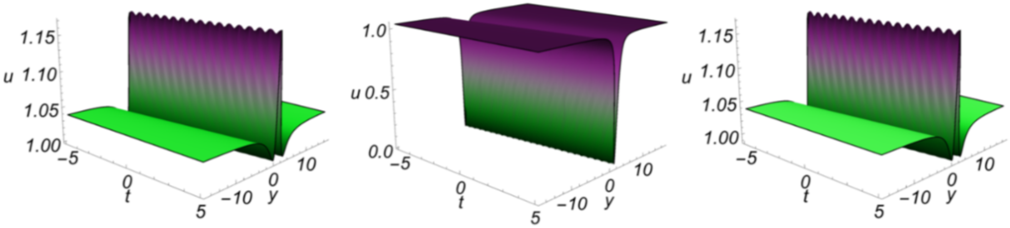

Figure 2: Singly localized bright and dark rational solitons in $y - t$.

First Order Rogue Wave

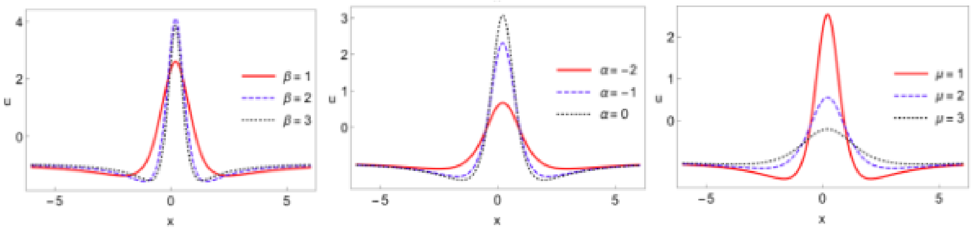

Figure 3: Role of arbitrary parameters β, α and μ.

- ⋆ Constraint conditions: $2a_1 + \alpha \neq 0$ and $1 + 2u_0\beta < 0$.
- ⋆ u_0, β, and γ: determine the type (bright or dark) rogue wave.
- ⋆ a_1 and α: directly proportional to the amplitude and tail-depth.
- ⋆ μ: inversely proportional/affecting the amplitude and tail-depth.
- ⋆ λ: shifts the position of rogue wave along the x axis.

Second Order Rogue Wave

Second-order rogue wave for $r = 1$ in Eq. (4a):

$$
\begin{aligned}
\mathcal{R}_2 &\Rightarrow R_2(x,\tau) + 2\lambda\tau F_1(x,\tau) + 2\mu x G_1(x,\tau) + (\lambda^2+\mu^2)R_0, \\
\mathcal{R}_2 &= (c_{0,0} + c_{0,2}\tau^2 + c_{0,4}\tau^4 + c_{0,6}\tau^6) + (c_{2,0} + c_{2,2}\tau^2 + c_{2,4}\tau^4)x^2 \\
&\quad +(c_{4,0} + c_{4,2}\tau^2)x^4 + x^6 + 2\lambda\tau(e_{0,0} + e_{0,2}\tau^2 + e_{2,0}x^2) \\
&\quad +2\mu x(h_{0,0} + h_{0,2}\tau^2 + h_{2,0}x^2) + (\lambda^2+\mu^2).
\end{aligned} \tag{7}
$$

Second Order Rogue Wave

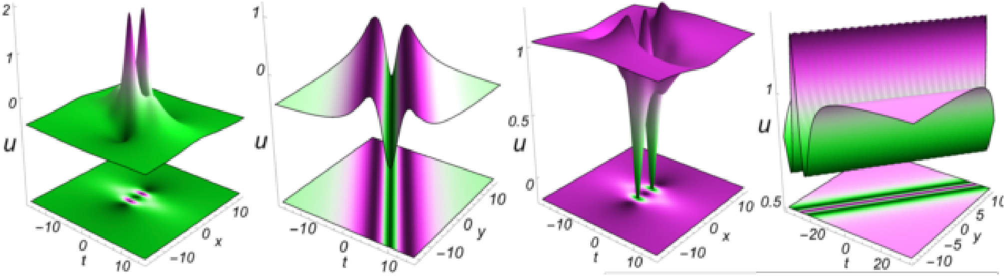

Figure 4: Doubly localized rogue waves in $x - t$ & singly-localized rational soliton in $y - t$.

Third Order Rogue Wave

Take $r = 2$ in (4a) to extract third-order rouge wave solution.

$$\mathcal{R}_3 \Rightarrow R_3(x,\tau) + 2\lambda\tau F_2(x,\tau) + 2\mu x G_2(x,\tau) + (\lambda^2 + \mu^2)R_1. \tag{8}$$

$$u = u_0 + \frac{6\gamma}{\beta}(ln\,\mathcal{R}_3)_{xx}. \tag{9}$$

Third Order Rogue Wave

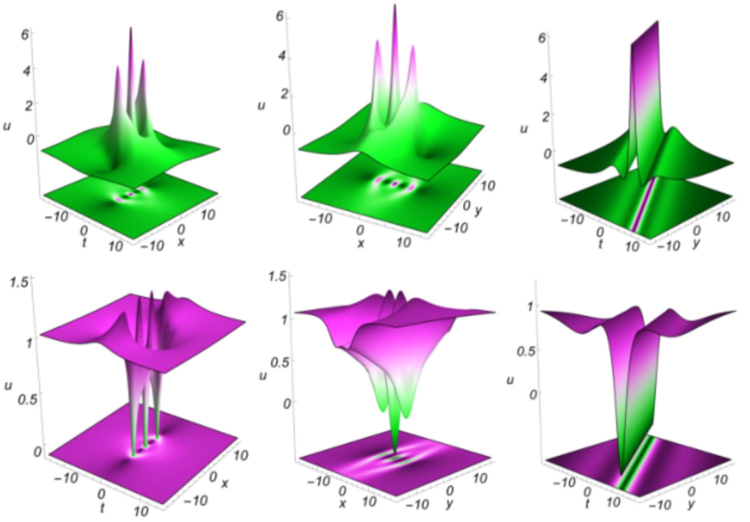

Figure 5: Doubly localized third-order bright and dark rogue waves in $x - t$ & $x - y$, and singly-localized bright and dark rational solitons in $y - t$.

Summary

- Considered a new integrable (2+1)-dimensional Boussinesq equation.
- Constructed higher-order rogue wave solutions by using Bell polynomial, Hirota's bilinearization and a generalized polynomial function.
- Explored the dynamics of these bright and dark localized structures and manipulation of their identities by tuning arbitrary parameters.
- The arbitrary parameters (u_0, β, γ, λ, α, μ, and a_1) help to control the amplitude/depth, width, tail-depth, and localization of the bright and dark rogue waves & rational solitons.
- Observed the evolution of W-shaped, M-shaped, and multi-peak rational solitons.
- The results will be encouraging to the studies on the rogue waves on other higher-dimensional systems.
- The study will be helpful to experimental investigations on the controlling mechanism of rogue waves in optical systems, atomic condensates, deep water oceanic waves, and other related coherent wave systems.

Reference

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Acknowledgments

Sudhir Singh thanks MHRD and National Institute of Technology Tiruchirappalli for institute fellowship.

Thank You All !!!